Intro. Number Theory

Notation

Background

We will use a bit of number theory to construct:

- Key exchange protocols
- Digital signatures
- Public-key encryption

This module: crash course on relevant concepts

More info: read parts of Shoup’s book referenced
at end of module

Notation

From here on:

- N denotes a positive integer.
- p denote a prime.

Notation: $\mathbb{Z}_N = \{0,1,2,\ldots,N-1\}$

Can do addition and multiplication modulo N

Modular arithmetic

Examples: let N = 12

$9 + 8 = 5$ in $\mathbb{Z}_{12}$

$5 \times 7 = 11$ in $\mathbb{Z}_{12}$

$5 - 7 = 10$ in $\mathbb{Z}_{12}$

Arithmetic in $\mathbb{Z}_N$ works as you expect, e.g $x \cdot (y+z) = x \cdot y + x \cdot z$ in $\mathbb{Z}_N$

Greatest common divisor

Def: For ints. x,y: **gcd(x, y)** is the greatest common divisor of x,y

Example: gcd(12, 18) = 6 $\boxed{2}\times 12 \boxed{-1}\times 18 = 6$

Fact: for all ints. x,y there exist ints. a,b such that

$$\mathbf{a \cdot x + b \cdot y = gcd(x,y)}$$

a,b can be found efficiently using the extended Euclid alg.

If gcd(x,y)=1 we say that x and y are **relatively prime**

Modular inversion

Over the rationals, inverse of 2 is ½ . What about $\mathbb{Z}_N$?

<u>Def</u>: The **inverse** of x in $\mathbb{Z}_N$ is an element y in $\mathbb{Z}_N$ s.t. $x \cdot y = 1$ in $\mathbb{Z}_N$

y is denoted x^{-1} .

Example: let N be an odd integer. The inverse of 2 in $\mathbb{Z}_N$ is $\frac{N+1}{2}$

$$2 \cdot \left(\frac{N+1}{2}\right) = N+1 = 1 \text{ in } \mathbb{Z}_N$$

Modular inversion

Which elements have an inverse in $\mathbb{Z}_N$?

Lemma: x in $\mathbb{Z}_N$ has an inverse if and only if gcd(x,N) = 1

Proof:

$\gcd(x,N)=1 \Rightarrow \exists\, a,b:\ a\cdot x + b\cdot N = 1 \Longrightarrow a\cdot x = 1$ in $\mathbb{Z}_N$

$\Longrightarrow x^{-1} = a$ in $\mathbb{Z}_N$

$\gcd(x,N) > 1 \Rightarrow \forall a:\ \gcd(a\cdot x, N) > 1 \Rightarrow a\cdot x \neq 1$ in $\mathbb{Z}_N$

$\gcd(x,N)=2 \Longrightarrow \forall a:\ a\cdot x$ is even $\Longrightarrow \overbrace{a\cdot x}^{\text{even}} \neq \overbrace{b\cdot N+1}^{\text{odd}}$

More notation

Def: $\mathbb{Z}_N^*$ = (set of invertible elements in $\mathbb{Z}_N$) =

= { x$\in \mathbb{Z}_N$: gcd(x,N) = 1 }

Examples:

1. for prime p, $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\} = \{1, 2, \ldots, p-1\}$
2. $\mathbb{Z}_{12}^*$ = { 1, 5, 7, 11}

For x in $\mathbb{Z}_N^*$, can find x^{-1} using extended Euclid algorithm.

Solving modular linear equations

Solve: $\mathbf{a \cdot x + b = 0}$ **in** $\mathbb{Z}_N$

Solution: $\mathbf{x = -b \cdot a^{-1}}$ **in** $\mathbb{Z}_N$

Find a^{-1} in $\mathbb{Z}_N$ using extended Euclid. Run time: $O(\log^2 N)$

What about modular quadratic equations?

next segments

End of Segment

Intro. Number Theory

Fermat and Euler

Review

N denotes an n-bit positive integer.   p denotes a prime.

- $Z_N \quad = \quad \{ 0, 1, \ldots, N-1 \}$
- $(Z_N)^* \quad = \quad$ (set of invertible elements in Z_N) $=$

  $= \quad \{ \; x \in Z_N \; : \; \gcd(x,N) = 1 \; \}$

Can find inverses efficiently using Euclid alg.:   time = $O(n^2)$

Fermat's theorem (1640)

Thm: Let p be a prime

$$\forall\, x \in (Z_p)^* : \quad x^{p-1} = 1 \text{ in } Z_p$$

Example: p=5. $3^4 = 81 = 1$ in Z_5

So: $x \in (Z_p)^* \Rightarrow x \cdot x^{p-2} = 1 \Rightarrow x^{-1} = x^{p-2}$ in Z_p

another way to compute inverses, but less efficient than Euclid

Application: generating random primes

Suppose we want to generate a large random prime

say, prime p of length 1024 bits (i.e. $p \approx 2^{1024}$)

Step 1: choose a random integer $p \in [\ 2^{1024}\ ,\ 2^{1025}-1\ ]$

Step 2: test if $2^{p-1} = 1$ in Z_p

If so, output p and stop. If not, goto step 1 .

Simple algorithm (not the best). **Pr[p not prime] < 2^{-60}**

The structure of $(Z_p)^*$

Thm (Euler): $(Z_p)^*$ is a **cyclic group**, that is

$\exists\ g \in (Z_p)^*$ such that $\{1, g, g^2, g^3, \ldots, g^{p-2}\} = (Z_p)^*$

g is called a **generator** of $(Z_p)^*$

Example: p=7. $\{1, 3, 3^2, 3^3, 3^4, 3^5\} = \{1, 3, 2, 6, 4, 5\} = (Z_7)^*$

Not every elem. is a generator: $\{1, 2, 2^2, 2^3, 2^4, 2^5\} = \{1, 2, 4\}$

Order

For $g \in (Z_p)^*$ the set $\{1, g, g^2, g^3, \ldots\}$ is called

the **group generated by g**, denoted $<g>$

Def: the **order** of $g \in (Z_p)^*$ is the size of $<g>$

$\text{ord}_p(g) \;=\; |<g>| \;=\; (\text{smallest } a>0 \text{ s.t. } g^a = 1 \text{ in } Z_p)$

Examples: $\text{ord}_7(3) = 6$; $\text{ord}_7(2) = 3$; $\text{ord}_7(1) = 1$

Thm (Lagrange): $\forall g \in (Z_p)^*$: **$\text{ord}_p(g)$** divides $p-1$

Euler's generalization of Fermat (1736)

Def: For an integer N define $\varphi(N) = |(Z_N)^*|$ (Euler's φ func.)

Examples: $\varphi(12) = |\{1,5,7,11\}| = 4$; $\varphi(p) = p-1$

For $N=p \cdot q$: $\varphi(N) = N-p-q+1 = (p-1)(q-1)$

Thm (Euler): $\forall\, x \in (Z_N)^*$: $x^{\varphi(N)} = 1$ in Z_N

Example: $5^{\varphi(12)} = 5^4 = 625 = 1$ in Z_{12}

Generalization of Fermat. Basis of the RSA cryptosystem

End of Segment

Intro. Number Theory

Modular e'th roots

Modular e'th roots

We know how to solve modular **linear** equations:

$\mathbf{a \cdot x + b = 0}$ in Z_N Solution: $\mathbf{x = -b \cdot a^{-1}}$ in Z_N

What about higher degree polynomials?

Example: let p be a prime and $c \in Z_p$. Can we solve:

$$x^2 - c = 0 \quad , \quad y^3 - c = 0 \quad , \quad z^{37} - c = 0 \quad \text{in} \quad Z_p$$

Modular e'th roots

Let p be a prime and $c \in Z_p$.

Def: $x \in Z_p$ s.t. $x^e = c$ in Z_p is called an **e'th root** of c .

$6^3 = 216 = 7$ in $\mathbb{Z}_{11}$

Examples: $7^{1/3} = 6$ in $\mathbb{Z}_{11}$

$3^{1/2} = 5$ in $\mathbb{Z}_{11}$

$1^{1/3} = 1$ in $\mathbb{Z}_{11}$

$2^{1/2}$ does not exist in $\mathbb{Z}_{11}$

The easy case

When does $c^{1/e}$ **in** Z_p exist? Can we compute it efficiently?

The easy case: suppose $\gcd(e, p-1) = 1$

Then for all c in $(Z_p)^*$: $c^{1/e}$ exists in Z_p and is easy to find.

Proof: let $d = e^{-1}$ **in** Z_{p-1}. Then $\boxed{c^{1/e} = c^d \text{ in } \mathbb{Z}_p}$

$d \cdot e = 1$ in Z_{p-1} $\Rightarrow \exists k \in \mathbb{Z} : d \cdot e = k \cdot (p-1) + 1 \Longrightarrow$

$\Longrightarrow (c^d)^e = c^{d \cdot e} = c^{k \cdot (p-1) + 1} = [c^{p-1}]^k \cdot c = c$ in $\mathbb{Z}_p$

The case e=2: square roots

If p is an odd prime then gcd(2, p-1) ≠ 1

$$\begin{tikzcd} x \arrow[dr] & & -x \arrow[dl] \\ & x^2 & \end{tikzcd}$$

Fact: in $\mathbb{Z}_p^*$, $x \longrightarrow x^2$ is a 2-to-1 function

Example: in $\mathbb{Z}_{11}^*$:

$$\begin{tikzcd}[column sep=small] 1 \arrow[dr] & & 10 \arrow[dl] & 2 \arrow[dr] & & 9 \arrow[dl] & 3 \arrow[dr] & & 8 \arrow[dl] & 4 \arrow[dr] & & 7 \arrow[dl] & 5 \arrow[dr] & & 6 \arrow[dl] \\ & 1 & & & 4 & & & 9 & & & 5 & & & 3 & \end{tikzcd}$$

Def: x in $\mathbb{Z}_p$ is a **quadratic residue** (Q.R.) if it has a square root in $\mathbb{Z}_p$

p odd prime ⇒ the # of Q.R. in $\mathbb{Z}_p$ is (p-1)/2 + 1

Euler's theorem

Thm: x in $(Z_p)^*$ is a Q.R. $\iff$ $x^{(p-1)/2} = 1$ in Z_p (p odd prime)

Example:

in $\mathbb{Z}_{11}$:	1^5,	2^5,	3^5,	4^5,	5^5,	6^5,	7^5,	8^5,	9^5,	10^5
=	1	-1	1	1	1,	-1,	-1,	-1,	1,	-1

Note: $x \neq 0 \Rightarrow x^{(p-1)/2} = \left(x^{p-1}\right)^{1/2} = 1^{1/2} \in \{1, -1\}$ in Z_p

Def: $x^{(p-1)/2}$ is called the **Legendre Symbol** of x over p (1798)

Computing square roots mod p

Suppose $p = 3 \pmod 4$

Lemma: if $c \in (Z_p)^*$ is Q.R. then $\sqrt{c} = c^{(p+1)/4}$ in Z_p

Proof: $\left[c^{\frac{p+1}{4}}\right]^2 = c^{\frac{p+1}{2}} = \underbrace{c^{\frac{p-1}{2}}}_{=1} \cdot c = c$ in $\mathbb{Z}_p$

When $p = 1 \pmod 4$, can also be done efficiently, but a bit harder

run time $\approx O(\log^3 p)$

Solving quadratic equations mod p

Solve: $a \cdot x^2 + b \cdot x + c = 0$ in Z_p

Solution: $x = \left(-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}\right) / 2a$ in Z_p

- Find $(2a)^{-1}$ in Z_p using extended Euclid.
- Find square root of $b^2 - 4 \cdot a \cdot c$ in Z_p (if one exists)

  using a square root algorithm

Computing e'th roots mod N ??

Let N be a composite number and e>1

When does **$c^{1/e}$ in Z_N** exist? Can we compute it efficiently?

Answering these questions requires the factorization of N

(as far as we know)

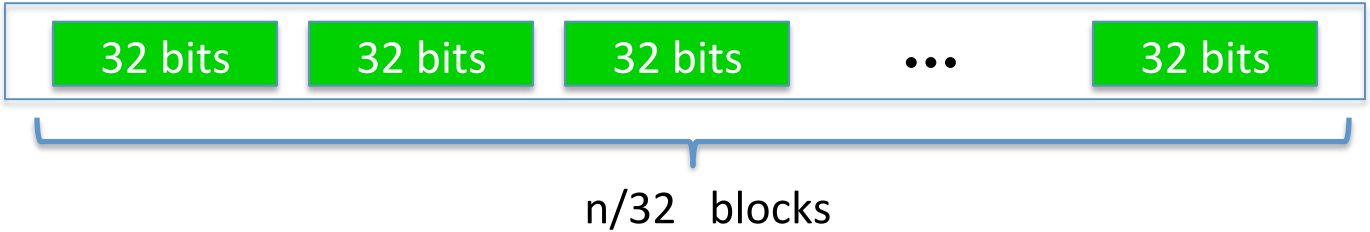
End of Segment

Intro. Number Theory

Arithmetic algorithms

Representing bignums

Representing an n-bit integer (e.g. n=2048) on a 64-bit machine



Note: some processors have 128-bit registers (or more) and support multiplication on them

Arithmetic

Given: two n-bit integers

- **Addition and subtraction**: linear time $O(n)$
- **Multiplication**: naively $O(n^2)$. Karatsuba (1960): $O(n^{\overset{\log_2 3}{\overset{\downarrow}{1.585}}})$

  Basic idea: $(2^b x_2 + x_1) \times (2^b y_2 + y_1)$ with 3 mults.

  Best (asymptotic) algorithm: about $O(n \cdot \log n)$.
- **Division with remainder**: $O(n^2)$.

Exponentiation

Finite cyclic group G (for example G = $\mathbb{Z}_p^*$)

Goal: given g in G and x compute g^x

Example: suppose $x = 53 = (110101)_2 = 32+16+4+1$

Then: $g^{53} = g^{32+16+4+1} = g^{32} \cdot g^{16} \cdot g^{4} \cdot g^{1}$

$$\begin{tikzcd}
g \arrow[r] \arrow[rrrrrr, bend left] & g^{2} \arrow[r] & g^{4} \arrow[r] \arrow[rrrr, bend left] & g^{8} \arrow[r] & g^{16} \arrow[r] \arrow[rr, bend left] & g^{32} \arrow[r, bend left] & g^{53}
\end{tikzcd}$$

The repeated squaring alg.

Input: g in G and x>0 ; **Output**: g^x

write $x = (x_n\, x_{n-1} \ldots x_2\, x_1\, x_0)_2$

```
$y \leftarrow g$ ,  $z \leftarrow 1$
for i = 0 to n do:
    if (x[i] == 1):    $z \leftarrow z \cdot y$
    $y \leftarrow y^2$
output z
```

example: g^{53}

y	z
g^2	g
g^4	g
g^8	g^5
g^{16}	g^5
g^{32}	g^{21}
g^{64}	$\mathbf{g^{53}}$

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Running times

Given n-bit int. N:

- **Addition and subtraction in $\mathbf{Z_N}$**: linear time $T_+ = O(n)$
- **Modular multiplication in $\mathbf{Z_N}$**: naively $T_\times = O(n^2)$
- **Modular exponentiation in $\mathbf{Z_N}$ ($\mathbf{g^x}$)**:

$$O\Big((\log x) \cdot T_\times \Big) \quad \leq \quad O\Big((\log x) \cdot n^2 \Big) \quad \leq \quad O(\, n^3 \,)$$

End of Segment

Intro. Number Theory

Intractable problems

Easy problems

- Given composite N and x in Z_N find x^{-1} in Z_N

- Given prime p and polynomial $f(x)$ in $Z_p[x]$

  find x in Z_p s.t. $f(x) = 0$ in Z_p (if one exists)

  Running time is linear in deg(f) .

… but many problems are difficult

Intractable problems with primes

Fix a prime p>2 and g in $(Z_p)^*$ of order q.

Consider the function: **$x \longmapsto g^x$ in Z_p**

Now, consider the inverse function:

$\mathrm{Dlog}_g\,(g^x) = x$ where x in $\{0, \ldots, q\text{-}2\}$

Example:

in $\mathbb{Z}_{11}$:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10
$\mathrm{Dlog}_2(\cdot)$:	0,	1,	8,	2,	4,	9,	7,	3,	6,	5

DLOG: more generally

Let **G** be a finite cyclic group and **g** a generator of G

$G = \{ 1, g, g^2, g^3, \ldots, g^{q-1} \}$ (q is called the order of G)

<u>Def</u>: We say that **DLOG is hard in G** if for all efficient alg. A:

$$\Pr_{g \longleftarrow G,\ x \longleftarrow Z_q} \big[A(G, q, g, g^x) = x \big] < \text{negligible}$$

Example candidates:

(1) $(Z_p)^*$ for large p, (2) Elliptic curve groups mod p

Computing Dlog in $(Z_p)^*$ (n-bit prime p)

Best known algorithm (GNFS): run time exp($\tilde{O}(\sqrt[3]{n})$)

cipher key size	modulus size	Elliptic Curve group size
80 bits	1024 bits	160 bits
128 bits	3072 bits	256 bits
256 bits (AES)	**15360** bits	512 bits

As a result: slow transition away from (mod p) to elliptic curves

An application: collision resistance

Choose a group G where Dlog is hard (e.g. $(Z_p)^*$ for large p)

Let $q = |G|$ be a prime. Choose generators g, h of G

For $x,y \in \{1,\ldots,q\}$ define $\boxed{H(x,y) = g^x \cdot h^y}$ in G

Lemma: finding collision for H(.,.) is as hard as computing $\mathrm{Dlog}_g(h)$

Proof: Suppose we are given a collision $H(x_0,y_0) = H(x_1,y_1)$

then $g^{x_0} \cdot h^{y_0} = g^{x_1} \cdot h^{y_1} \Rightarrow g^{x_0-x_1} = h^{y_1-y_0} \Rightarrow h = g^{x_0-x_1/\overset{\neq 0}{y_1-y_0}}$

Intractable problems with composites

Consider the set of integers: (e.g. for n=1024)

$$\mathbb{Z}_{(2)}(n) \;:=\; \{\, N = p \cdot q \;\text{ where } p,q \text{ are n-bit primes} \,\}$$

Problem 1: Factor a random N in $\mathbb{Z}_{(2)}(n)$ (e.g. for n=1024)

Problem 2: Given a polynomial **f(x)** where degree(f) > 1

and a random N in $\mathbb{Z}_{(2)}(n)$

find x in $\mathbb{Z}_N$ s.t. f(x) = 0 in $\mathbb{Z}_N$

The factoring problem

Gauss (1805): *"The problem of distinguishing prime numbers from composite numbers and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic."*

Best known alg. (NFS): run time exp($\tilde{O}(\sqrt[3]{n})$) for n-bit integer

Current world record: **RSA-768** (232 digits)

- Work: two years on hundreds of machines
- Factoring a 1024-bit integer: about 1000 times harder

⇒ likely possible this decade

Further reading

- A Computational Introduction to Number Theory and Algebra, V. Shoup, 2008 (V2), Chapter 1-4, 11, 12

Available at **//shoup.net/ntb/ntb-v2.pdf**

End of Segment